INVARIANTS OF THE REAL LINE AFFINE GROUP ON THE SPACE OF POLYNOMIALS [1]

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In this paper we study orbits and invariants of the group $x \mapsto \alpha x+\beta$ acting by conjugations on the spaces of polynomials $f(x)$.

1 Orbits of the affine group on the space of polynomials

Let V_n be the space of polynomials $f(x)$ of degree $\leqslant n$ over the field $\mathbb{R}$:

$$f(x) = a_0 + a_1 x + \ldots + a_n x^n, \tag{1.1}$$

where $a_0, a_1, \ldots, a_n \in \mathbb{R}$ and the variable x ranges the space $\mathbb{R}$. The space V_n has dimension $n+1$. Let us denote by V_n^+, V_n^-, V_n^0 the subsets of V_n consisting of polynomials $f(x)$ with $a_n > 0$, $a_n < 0$, $a_n = 0$, respectively. The subset V_n^0 can be identified with V_{n-1} in the natural way.

Therefore, we have a filtration

$$\mathbb{R} = V_0 \subset V_1 \subset V_2 \subset \ldots \subset V_n \subset \ldots \tag{1.2}$$

Let G be the connected group of affine transformations φ of the real line $\mathbb{R}$:

$$x \mapsto \varphi(x) = \alpha x + \beta,$$

where $\alpha, \beta \in \mathbb{R}$, $\alpha > 0$. It acts on the space V_n of polynomials $f(x)$ as follows:

$$T(\varphi) f = \varphi^{-1} \circ f \circ \varphi, \tag{1.3}$$

or

$$(T(\varphi)f)(x) = \frac{1}{\alpha} f(\alpha x + \beta) - \frac{\beta}{\alpha}.$$

Since φ is determined by parameters α, β, we sometimes write $T(\alpha, \beta)$ instead of $T(\varphi)$.

Let us assign to the polynomial (1.1) a column of its coefficients $a = (a_0, a_1, \ldots, a_n)$. Then the transformation $T(\varphi)$ is the following affine transformation of the space $V_n = \mathbb{R}^{n+1}$:

$$a \mapsto \widetilde{a} = A(\varphi) a + b(\varphi),$$

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where $A(\varphi)$ is a upper triangular matrix with entries

$$A_{km}(\varphi) = \binom{m}{k}\alpha^{k-1}\beta^{m}, \tag{1.4}$$

where $k, m \in \{0, 1, \dots, n\}$, so that $A_{km} = 0$ for $k < m$, and $b(\varphi)$ is a column $(-\beta/\alpha, 0, \dots, 0)$. For example, for $n = 4$ we have

$$\begin{pmatrix} \tilde{a}_0 \\ \tilde{a}_1 \\ \tilde{a}_2 \\ \tilde{a}_3 \\ \tilde{a}_4 \end{pmatrix} = \begin{pmatrix} 1/\alpha & \beta/\alpha & \beta^2/\alpha & \beta^3/\alpha & \beta^4/\alpha \\ 0 & 1 & 2\beta & 3\beta^2 & 4\beta^3 \\ 0 & 0 & \alpha & 3\alpha\beta & 6\alpha\beta^2 \\ 0 & 0 & 0 & \alpha^2 & 4\alpha^2\beta \\ 0 & 0 & 0 & 0 & \alpha^3 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} + \begin{pmatrix} -\beta/\alpha \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

We see from (1.4) that the group G preserves V_n^+, V_n^- and $V_n^0 = V_{n-1}$, so that the group G preserves the filtration (1.2).

Consider G-orbits in V_n.

For $n = 1$, there is a fixed point $(0, 1)$, i.e. the polynomial $f(x) = x$, and G-orbits are: lines $a_1 = c$ with $c \neq 1$ and two rays $(a_0, 1)$, $a_0 > 0$ and $(a_0, 1)$, $a_0 < 0$.

Now let $n \geqslant 2$.

Theorem 1.1 *The group G acts simply transitively on its orbits in V_n^+ and V_n^-.*

Proof. The Lie algebra of the group G consists of linear combinations of two differential operators:

$$L_1 = x\frac{\partial}{\partial x}, \quad L_2 = \frac{\partial}{\partial x}.$$

These operators are mapped to the following vector fields in V_n:

$$\xi_1 = (-a_0, 0, a_2, 2a_3, \dots, (n-1)a_n), \tag{1.5}$$

$$\xi_2 = (a_1 - 1, 2a_2, 3a_3, \dots, na_n, 0), \tag{1.6}$$

The matrix formed by these vectors (1.5), (1.6) has rank 2 for $a \in V_n^{\pm}$ and $n \geqslant 2$. □

2 Invariants of the affine group on the space of polynomials

Let us write invariants of the action (1.3) of the group G on the space V_n. Since G preserves the filtration (1.2), we can consider only $V_n^{\pm}$. Moreover, we can restrict ourselves to V_n^+. Indeed, let σ be a map of V_n assigning to a polynomial $f(x)$ the polynomial $-f(x)$, i.e. $\sigma = -\mathrm{id}$. Then

$$T(\alpha, \beta)\sigma = \sigma T(\alpha, -\beta).$$

Thus, let $a_n > 0$. Since dimension of any G-orbit is equal to 2 and dimension of the space V_n is equal to $n + 1$, we have to write $n - 1$ invariants. They are shown by the following theorem.

Theorem 2.1 *Invariants of the group G on the set V_n^+ are the following $n-1$ functions $\Phi_1, \ldots, \Phi_{n-1}$ of coefficients $a_0, a_1, \ldots, a_n$ of polynomials f:*

$$\begin{aligned}
\Phi_k &= a_n^{-\frac{(n-2)(k+1)}{n-1}} \left\{ (-1)^k k \binom{n}{k+1} a_{n-1}^{k+1} + \right. \\
&+ \left. \sum_{i=0}^{k} (-1)^{k-i} n^{i+1} \binom{n-1-i}{k-i} a_n^i a_{n-1}^{k-i} a_{n-i-1} \right\}, k = 1, \ldots, n-2, \\
\Phi_{n-1} &= a_n^{-\frac{(n-2)n}{n-1}} \left\{ (-1)^{n-1}(n-1) a_n^{n-1} + \right. \\
&+ \left. \sum_{i=1}^{n-1} (-1)^{n-1-i} n^{i+1} a_n^i a_{n-1}^{n-1-i} a_{n-i-1} + n^{n-1} a_n^{n-1} a_{n-1} \right\}
\end{aligned}$$

We omit the proof because it is rather cumbersome.

For example, for V_2^+ we have one invariant:

$$\Phi_1 = -a_1^2 + 2a_1 + 4a_0 a_2$$

For V_3^+ we have 2 invariants:

$$\begin{aligned}
\Phi_1 &= a_3^{-1}(-3a_2^2 + 9a_3 a_1), \\
\Phi_2 &= a_3^{-3/2}(9a_3^2 a_2 + 2a_2^3 - 9a_3 a_2 a_1 + 27 a_3^2 a_0)
\end{aligned}$$

For V_4^+ we have 3 invariants:

$$\begin{aligned}
\Phi_1 &= a_3^{-4/3}(-6a_3^2 + 16a_4 a_2), \\
\Phi_2 &= a_4^{-2}(8a_3^3 - 32a_4 a_3 a_2 + 64 a_4^2 a_1), \\
\Phi_3 &= a_4^{-8/3}(-3a_3^4 + 16 a_4 a_3^2 a_2 - 64 a_4^2 a_3 a_1 + 256 a_4^3 a_0 + 64 a_4 a_3)
\end{aligned}$$

Orbits of the group G in V_n^+ are connected parts of intersections of level surfaces of these invariants:

$$\Phi_1 = C_1, \ldots, \Phi_{n-1} = C_{n-1}.$$

It would be very interesting to find explicit expressions of Φ_k in terms of roots of $f^{(n-k-1)}(x)$. We are able to do it for $k = 1$ and $k = 2$.

The invariant Φ_1 for $n \geqslant 3$ is up to a factor the discriminant of the derivative $f^{(n-2)}(x)$. Indeed, this derivative is

$$f^{(n-2)}(x) = \frac{n!}{2!} a_n \left\{ x^2 + b_1 x + b_2 \right\},$$

where

$$b_1 = \frac{2}{n} \cdot \frac{a_{n-1}}{a_n}, \quad b_2 = \frac{2}{n(n-1)} \cdot \frac{a_{n-2}}{a_n},$$

then the invariant Φ_1 is

$$\Phi_1 = -\left(\frac{n}{2}\right)^2 \cdot \binom{n}{2} a_n^{2/n-1}(b_1^2 - 4b_2).$$

Let y_1, y_2 be the roots of the polynomial $x^2 + b_1 x + b_2$. Then

$$b_1^2 - 4b_2 = (y_1 - y_2)^2.$$

The invariant Φ_2 for $n \geqslant 4$ is expressed in terms of the derivative $f^{(n-3)}(x)$ as follows. This derivative is

$$f^{(n-3)}(x) = \frac{n!}{3!} a_n (x^3 + c_1 x^2 + c_2 x + c_3),$$

where

$$c_1 = \frac{3}{n} \cdot \frac{a_{n-1}}{a_n}, \quad b_2 = \frac{3!}{n(n-1)} \cdot \frac{a_{n-2}}{a_n}, \quad c_3 = \frac{3!}{n(n-1)(n-2)} \cdot \frac{a_{n-3}}{a_n}.$$

Then the invariant Φ_2 is

$$\Phi_2 = \left(\frac{n}{3}\right)^3 \cdot \binom{n}{3} a_n^{\frac{3}{n-1}}(2c_1^3 - 9c_1 c_2 + 27c_3).$$

Let z_1, z_2, z_3 be the roots of the polynomial $x^3 + c_1 x^2 + c_2 x + c_3$. Then the polynomial $\varphi = 2c_1^3 - 9c_1 c_2 + 27c_3$ can be written in terms of differences of roots z_1, z_2, z_3 as follows:

$$\begin{aligned} \varphi = & - (z_1 - z_2)^2 (z_1 + z_2 - 2z_3) - \\ & - (z_1 - z_3)^2 (z_1 + z_3 - 2z_2) - \\ & - (z_2 - z_3)^2 (z_2 + z_3 - 2z_1) \end{aligned}$$